Lecture 18 (TMA4044)

Green's & Stoke's theorems

Outline

Part I:
- Revisiting the FTC
- Green's theorem (statement & intuition)
- Stoke's theorem
- Examples & applications (Maxwell-Faraday's law, area calculations)

Part II:
- Proof of Green's
- Proof of Stoke's (by reduction to Green's)

~~Proofs will be available on the youtube-channel?~~

Recall: Fundamental thm of calculus states

$f(x)$ a cont. diff. function on an intervall $I = [a,b]$. Then:

$$\int_a^b \frac{df}{dx}\, dx = f(b) - f(a)$$

Now notice that this can be written as

$$\int_I df = \overbrace{\int_{\partial I} f}^{= f(b) - f(a)}$$

interval with orientation

$a \longrightarrow b$

differential of f = infinitesimal increment

Boundary of I = endpoints b & a.

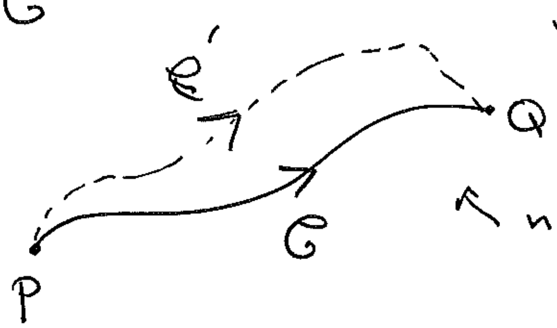
N.B. On the right handside we must remember the orientation to set the correct sign!

Recall similar(?) fundamental thm for line integrals:

smooth scalar potential

$$\int_{\mathcal{C}} \underbrace{\nabla \phi(x,y,z)}_{=\mathbb{F}(x,y,z)} \cdot d\mathbf{s} = \underbrace{\phi(P) - \phi(Q)}_{\text{independent of the path!}}$$

conservative vector field!



nice curve (piecewise continuous, simple)

we can write this as

$$\int_{\mathcal{C}} d\phi = \int_{\partial \mathcal{C}} \phi$$

chain rule:

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = \nabla \phi \cdot d\mathbf{r}$$

In fact, one can generalize these formula to surfaces:

$$\int_{S} \text{"derivative on } \omega \text{"} = \int_{\partial S} \omega$$

surface ~~some manifold (curve, surface, etc.~~)

border of S — a curve

Hence, Green's thm can also be written as

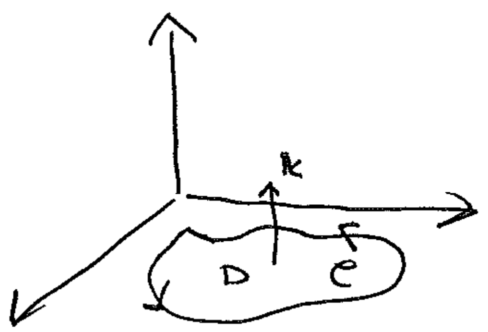
$$\iint_D (\text{curl}(\mathbb{F}) \cdot \mathbb{k})\, dA = \oint_{\mathcal{C}} \mathbb{F} \cdot d\mathbb{r}$$

<u>Quiz</u>: $\oint_{\mathcal{C}} \mathbb{F} \cdot d\mathbb{r} > 0$ if the vector field has an overall counterclockwise rotation around $\mathcal{C}$.

True or False?

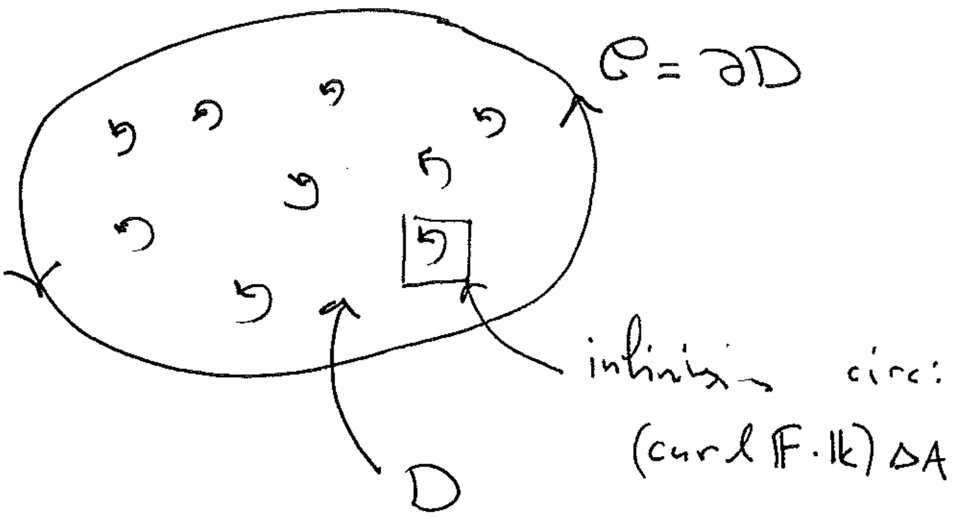
<u>Remarks</u>:

* $\mathbb{k}$ is th upward normal



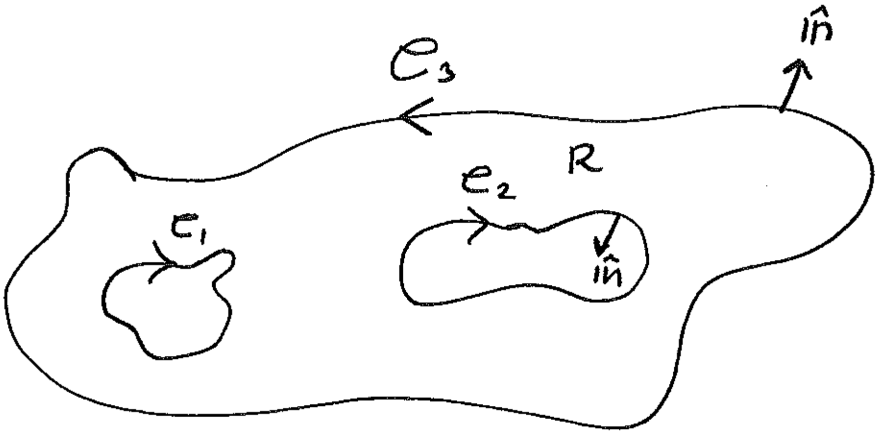
* The orientation on $\mathcal{C}$ is induced by th right hand rule.

* Physical intuition



Green's thm states that the sum of "microsopic circulation" of $\mathbb{F}$ over D equals total circulation of $\mathbb{F}$ along the boundary:

Subtlety: non-simply connected regions



Green's theorem still holds with proper orientation of the boundary $\partial R = C_1 + C_2 + C_3$

$$\iint_R (\text{curl}\,\mathbb{F})\cdot\hat{n}\, dA = \int_{C_1} \mathbb{F}\cdot d\mathbb{r} + \int_{C_2} \mathbb{F}\cdot d\mathbb{r} + \int_{C_3} \mathbb{F}\cdot d\mathbb{r}$$

Consider $\mathbb{F}$ such that $\nabla\times\mathbb{F} = 0$.
Naively we would say this implies $\mathbb{F} = \nabla\phi$ (conservative).
This is only true if R is simply connected!

Ex: $\mathbb{F} = -\frac{y}{r}\mathbb{i} + \frac{x}{r}\mathbb{j}$ defined on punctured disc $0 < r^2 \leq a^2$

we have $\text{curl}\,\mathbb{F}\cdot\mathbb{k} = 0 \Rightarrow \iint_R (\text{curl}\,\mathbb{F})\cdot\mathbb{k}\, dA = 0$

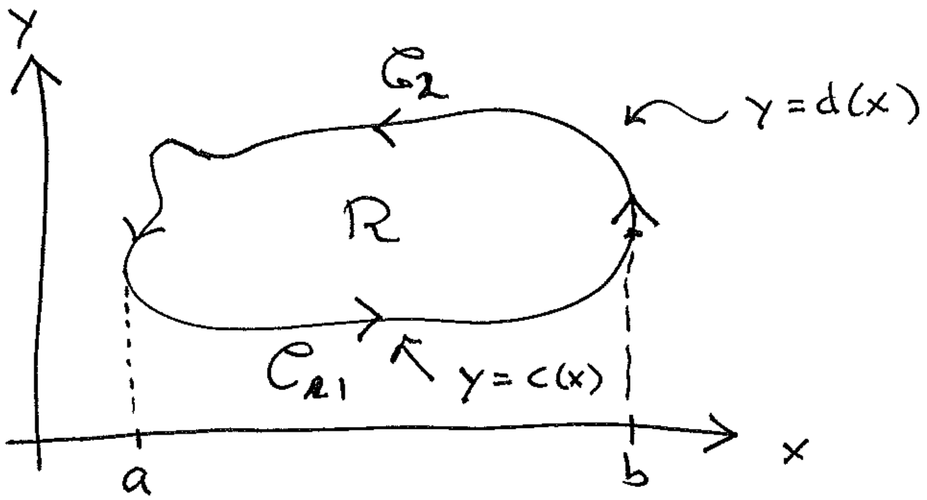
But $\oint_C \mathbb{F}\cdot d\mathbb{r} = 2\pi$.

Proof of Green's theorem

Green's theorem in xy-plane:

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \oint_{\mathcal{C}} \mathbb{F} \cdot d\mathbb{r}$$

Proof Recall that each closed domain R can be divided into "x-simple" and "y-simple" parts.



$$\Rightarrow \iint_R \frac{\partial F_1}{\partial y}\, dx\,dy = \int_a^b dx \int_{c(x)}^{d(x)} \frac{\partial F_1}{\partial y}\, dy$$

FTC

$$= \int_a^b \Big(F_1(x, d(x)) - F_1(x, c(x)) \Big)\, dx$$

Now focus on the line integral: (20)

$$\int_{C} F_1(x,y)\,dx = \int_{C_1} F_1(x, c(x))\,dx$$

$$+ \int_{C_2} F_1(x, d(x))\,dx$$

$$= \int_a^b \left[F_1(x, c(x)) - F_1(x, d(x)) \right] dx$$

$$\overset{*}{=} - \iint_R \frac{\partial F_1}{\partial y}\,dx\,dy$$

$$\Rightarrow - \iint_R \frac{\partial F_1}{\partial y}\,dx\,dy = \oint_C F_1(x,y)\,dx$$

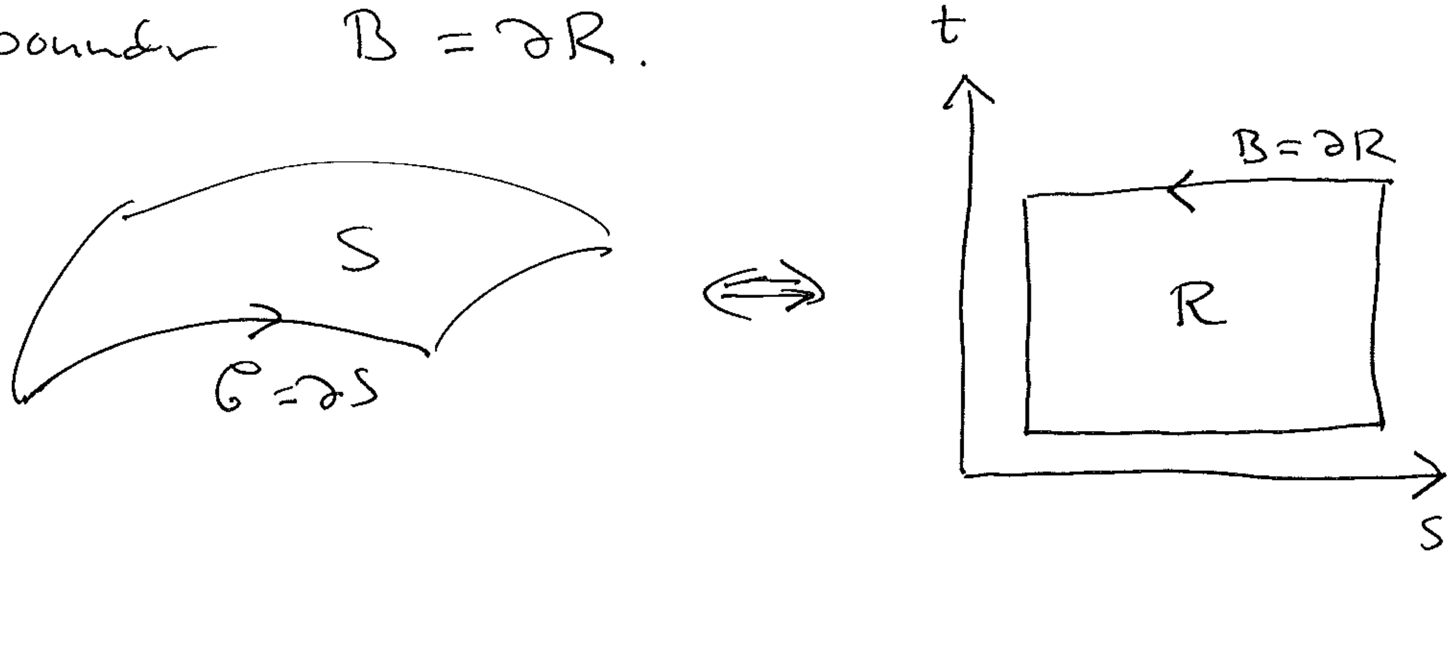
Proof of Stoke's thm

$$\iint_S \mathrm{curl}(\mathbb{F}) \cdot d\mathbb{S} = \int_{C = \partial S} \mathbb{F} \cdot d\mathbb{r}$$

Parametrize the surface S in terms of s & t such that

$$\mathbb{r} = \mathbb{r}(s,t)$$

Then S corresponds to a region R in the (s,t)-plane and the curve $C = \partial S$ corresponds to the boundar $B = \partial R$.



We can then convert the line integral around $\mathcal{C}$ into a line integral around B:

$$\oint_{\mathcal{C}} \mathbb{F} \cdot d\mathbb{r} = \oint_{B} \mathbb{F} \cdot \frac{\partial \mathbb{r}}{\partial s}\, ds + \mathbb{F} \cdot \frac{\partial \mathbb{r}}{\partial t}\, dt$$

If we now define a vector field on the (s,t)-plane by

$$\mathbb{G} = \mathbb{G}(s,t) \quad \text{with}$$

$$G_1 = \mathbb{F} \cdot \frac{\partial \mathbb{r}}{\partial s} \qquad G_2 = \mathbb{F} \cdot \frac{\partial \mathbb{r}}{\partial t}$$

we can write the line integral as

$$\oint_{\mathcal{C}} \mathbb{F} \cdot d\mathbb{r} = \int_{B} G_1\, ds + G_2\, dt$$

$$= \int_{B} \mathbb{G} \cdot d\mathbb{s}$$

Now consider the flux integral in the same parametrization:

$$\iint_S \text{curl}(\mathbb{F}) \cdot d\mathbb{S} = \iint_R \text{curl}\,\mathbb{F} \cdot \left(\frac{\partial \mathbb{r}}{\partial s} \times \frac{\partial \mathbb{r}}{\partial t}\right) \underbrace{dA}_{= ds\,dt}$$

Since $\text{curl}\,\mathbb{F} \cdot \left(\frac{\partial \mathbb{r}}{\partial s} \times \frac{\partial \mathbb{r}}{\partial t}\right)$

$$= \frac{\partial G_2}{\partial s} - \frac{\partial G_1}{\partial t}$$

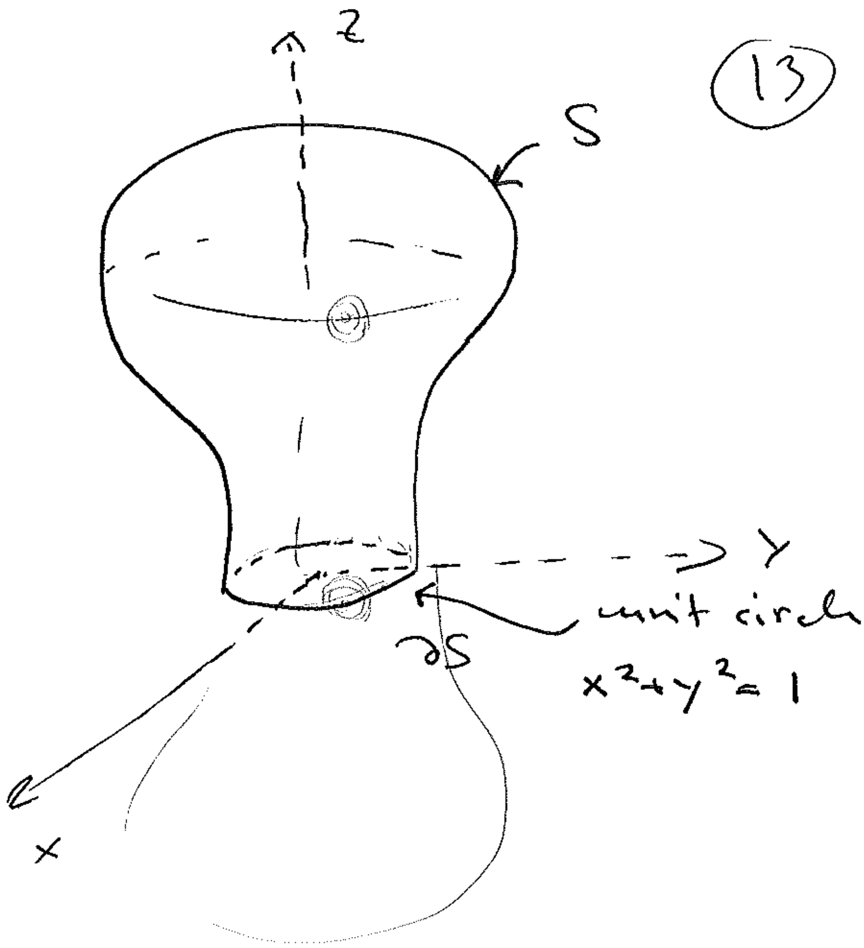
we can write

$$\iint_S \text{curl}(\mathbb{F}) \cdot d\mathbb{S} = \iint_R \left(\frac{\partial G_2}{\partial s} - \frac{\partial G_1}{\partial t}\right) ds\,dt$$

So stoke's theorem reduces to Green's theorem:

$$\iint_R \left(\frac{\partial G_2}{\partial s} - \frac{\partial G_1}{\partial t}\right) ds\,dt = \int_B \mathbb{G} \cdot d\mathbb{s}.$$

Example



vector field

$$\mathbb{F}(x,y,z) = e^{z^2-2z}\,\mathbb{i} + (\sin(xyz)+y+1)\,\mathbb{j} + e^{z^2}\sin z^2\,\mathbb{k}$$

Compute the integral $\iint_S \text{curl}\,\mathbb{F}\cdot d\mathbb{S}$ using Stoke's theorem.

Solution: Parametize the boundary ∂S:

$$\mathbb{r}(t) = \cos t\,\mathbb{i} + \sin t\,\mathbb{j} + 0.\mathbb{k}$$

$(0 \le t \le 2\pi)$ $\quad d\mathbb{r} = (-\sin t)\mathbb{i} + \cos t\,\mathbb{j}$

$$\Rightarrow \mathbb{F}(\mathbb{r}(t)) = \cos t\,\mathbb{i} + (1+\sin t)\,\mathbb{j} + 0.\mathbb{k}$$

$$\Rightarrow \iint_S \text{curl}(\mathbb{F})\cdot d\mathbb{S} \underset{\text{Stokes}}{=} \oint_{\partial S} \mathbb{F}\cdot d\mathbb{r} = \int_0^{2\pi} \cos t\,dt$$

$$= 0.$$

Maxwell - Faraday's law

Let $\mathbb{E}$ and $\mathbb{B}$ be electric and magnetic fields in $\mathbb{R}^3$.

The magnetic flux through an oriented surface S is given by:

$$\iint_S \mathbb{B}\cdot \underbrace{d\mathbb{S}}_{=\hat{n}\cdot dS}$$

Similarly, the circulation of an electric field around a simple closed curve $\mathcal{C}$ is:

$$\oint_{\mathcal{C}} \mathbb{E}\cdot ds$$

Faraday noticed that the circulation of the electric field around induces a change in the magnetic flux thru S with boundary $\mathcal{C}$ according to

$$\frac{d}{dt} \iint_S \mathbb{B} \cdot d\mathbb{S} = - \oint_{\mathcal{C} = \partial S} \mathbb{E} \cdot ds \quad \bigstar$$

Now use Stoke's thm to the RHS:

$$- \oint_{\mathcal{C} = \partial S} \mathbb{E} \cdot ds \overset{\text{Stoke}}{=} - \iint_S (\nabla \times \mathbb{E}) \cdot \hat{\mathbb{N}} \, dS$$

$$= - \iint_S (\nabla \times \mathbb{E}) \cdot d\mathbb{S}$$

We can then rewrite $\bigstar$ as:

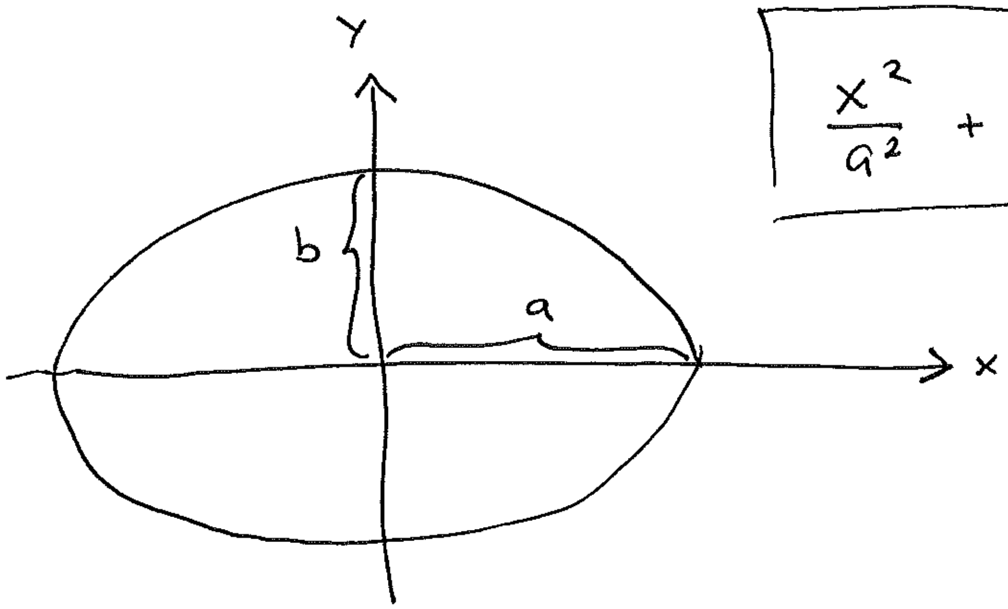
$$\iint_S - \frac{d}{dt} \mathbb{B} \cdot d\mathbb{S} = \iint_S (\nabla \times \mathbb{E}) \cdot d\mathbb{S}$$

Since S is an arbitrary surface we set:

$$\nabla \times \mathbb{E} = - \frac{d\mathbb{B}}{dt}$$

one of Maxwell's equations !!

Example: Area of an ellipse.



$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

How to compute the area?

If $a = b = 1$ we know that Area $= \pi$.

Use Green's thm:

$$\iint_D (\nabla \times \mathbb{F}) \cdot \mathbb{k} \, dA = \oint_{\partial D = C} \mathbb{F} \cdot d\mathbb{r}$$

General form of area of a domain D:

$$\text{Area} = \iint_D dA$$

One could evaluate this directly

But simpler to use Green!
Look for a vector field $\mathbb{F}(x,y)$ such that

$$(\nabla \times \mathbb{F}) \cdot \mathbb{k} = 1$$

One possible choice: $\mathbb{F}(x,y) = \frac{1}{2}(-y\,\mathbb{i} + x\,\mathbb{j})$

$\Rightarrow$ Express area as a line integral along the curve $\mathcal{C} = \partial D$ around D:

$$\text{area}(D) = \iint_D dA = \oint_{\mathcal{C}} \mathbb{F} \cdot d\mathbb{r}$$

$$= \frac{1}{2} \oint_{\mathcal{C}} x\,dy - y\,dx$$

Parametrize $\mathcal{C}$: $\begin{cases} x = a\cos\theta \\ y = b\sin\theta \end{cases}$ $\quad \underline{0 \le \theta \le 2\pi}$

$$\Rightarrow \text{area}(D) = \frac{1}{2}\int_0^{2\pi} \left[(a\cos\theta)(b\cos\theta) - (b\sin\theta)(-a\sin\theta)\right] d\theta$$

$$= \dots = \pi a b$$

<u>Try this!</u>